Public Announcements in Strategic Games with Arbitrary Strategy Sets

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Abstract. In van Benthem [2007] the concept of a public announcement is used to study the effect of the iterated elimination of strictly dominated strategies. We offer a simple generalisation of this approach to cover arbitrary strategic games and many optimality notions. We distinguish between announcements of optimality and announcements of rationality.

1 Introduction

One of the objectives of the theory of strategic games is to predict the choices players will make. This is usually done by studying iterated elimination of strategies that one assumes players will not choose. In this paper we investigate to what extent this elimination process can be explained using the concept of public announcements.

Public announcements were introduced into epistemic logic in Plaza [1989]. In van Benthem [2007] they were used to describe the process of iterated elimination of strictly dominated strategies in a finite game as a series of public announcements that transform an epistemic model of the players. In this paper we pursue this idea and obtain a generalization of van Benthem's results to arbitrary strategic games and to various different optimality operators that the players might employ to select their strategies.

The paper is organized as follows. In the next section we discuss the notions we shall rely on. Next, in Section 3 we introduce optimality properties that we discuss in the paper. We also explain the crucial distinction between 'local' and 'global' properties. Globality means that each subgame obtained by iterated elimination of strategies is analyzed *in the context* of the given initial game, while locality means that the context is the current subgame.

In Section 4, we consider public announcements of 'optimality'. We partially motivate van Benthem's use of what we call 'standard' models for games. In Section 5, we consider public announcements of 'rationality' which are based on players knowledge. This analysis shows the relevance of the concept of global optimality properties.

Unlike either of the two cited papers, we do not examine any of the interesting logical questions of axiomatisation or definability.

2 Preliminaries

In this section we introduce notions central to this note, and recall some basic results concerning them.

2.1 Operators

Consider a fixed complete lattice $(D, \subseteq)$ with the largest element $\top$. In what follows we use ordinals and denote them by α, β, γ. Given a, possibly transfinite, sequence $(G_\alpha)_{\alpha < \gamma}$ of elements of D we denote their meet by $\bigcap_{\alpha < \gamma} G_\alpha$.

Definition 1. *Let T be an operator on $(D, \subseteq)$, i.e., $T : D \to D$.*

- *We call T **contracting** if for all G, $T(G) \subseteq G$.*
- *We say that an element G is a **fixpoint** of T if $T(G) = G$.*
- *We define by transfinite induction a sequence of elements T^α of D, where α is an ordinal, as follows:*
  - $T^0 := \top$,
  - $T^{\alpha+1} := T(T^\alpha)$,
  - *for limit ordinals β, $T^\beta := \bigcap_{\alpha < \beta} T^\alpha$.*
- *We call the least α such that $T^{\alpha+1} = T^\alpha$ the **closure ordinal** of T and denote it by α_T. We call then T^{α_T} the **outcome of** (iterating) T and write it alternatively as T^∞.* □

So an outcome is a fixpoint reached by a transfinite iteration that starts with the largest element. In general, the outcome of an operator does not need to exist but we have the following obvious observation.

Note 1. Every contracting operator T on $(D, \subseteq)$ has an outcome, i.e., T^∞ is well-defined. □

2.2 Strategic games

Given n players $(n > 1)$ by a **strategic game** (in short, a **game**) we mean a sequence $(S_1, \ldots, S_n, p_1, \ldots, p_n)$, where for all $i \in \{1, \ldots, n\}$, S_i is the non-empty set of **strategies** available to player i, and p_i is the **payoff function** for the player i, so $p_i : S_1 \times \ldots \times S_n \to \mathcal{R}$, where $\mathcal{R}$ is the set of real numbers.

We denote the strategies of player i by s_i, possibly with some superscripts. Given $s \in S_1 \times \ldots \times S_n$ we denote the ith element of s by s_i, write sometimes s as (s_i, s_{-i}), and use the following standard notation:

- $s_{-i} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$,

- $S_{-i} := S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$.

In the remainder of the paper we assume an initial strategic game

$$H := (H_1, \ldots, H_n, p_1, \ldots, p_n).$$

A ***restriction*** of H is a sequence $(G_1, \ldots, G_n)$ such that $G_i \subseteq H_i$ for all $i \in \{1, \ldots, n\}$. We identify the restriction $(H_1, \ldots, H_n)$ with H. We shall focus on the complete lattice that consists of the set of all restrictions of the game H ordered by the componentwise set inclusion:

$$(G_1, \ldots, G_n) \subseteq (G'_1, \ldots, G'_n) \text{ iff } G_i \subseteq G'_i \text{ for all } i \in \{1, \ldots, n\}.$$

So H is the largest element in this lattice and $\bigcap_{\alpha < \gamma}$ is the customary set-theoretic operation on the restrictions.

Consider now a restriction $G := (G_1, \ldots, G_n)$ of H and two strategies s_i, s'_i from H_i (so *not necessarily* from G_i). We say that s_i ***is strictly dominated on*** G by s'_i (and write $s'_i \succ_G s_i$) if

$$\forall s_{-i} \in G_{-i} \; p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}),$$

and that s_i ***is weakly dominated on*** G by s'_i (and write $s'_i \succ^w_G s_i$) if

$$\forall s_{-i} \in G_{-i} p_i(s'_i, s_{-i}) \geq p_i(s_i, s_{-i}) \wedge \exists s_{-i} \in G_{-i} p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}).$$

Given a restriction $G' := (G'_1, \ldots, G'_n)$ of H, we say that the strategy s_i from H_i is a ***best response in*** G' ***to*** $s_{-i} \in G_{-i}$ if

$$\forall s'_i \in G'_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

2.3 Possibility correspondences

In this and the next subsection we recall the basic notions, following Battigalli and Bonanno [1999]. Fix a non-empty set Ω of ***states***. By an ***event*** we mean a subset of Ω.

A ***possibility correspondence*** is a mapping from Ω to the powerset $\mathcal{P}(\Omega)$ of Ω. The following three properties of a possibility correspondence P are relevant:

(i) for all ω, $P(\omega) \neq \emptyset$,
(ii) for all ω and ω', $\omega' \in P(\omega)$ implies $P(\omega') = P(\omega)$,
(iii) for all ω, $\omega \in P(\omega)$.

If the possibility correspondence satisfies properties (i)–(iii), we call it a ***knowledge correspondence***.[4] Note that each knowledge correspondence P yields a partition $\{P(\omega) \mid \omega \in \Omega\}$ of Ω.

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[4] In the modal logic terminology a knowledge correspondence is a frame for the modal logic S5, see, e.g. Blackburn, de Rijke and Venema [2001].

2.4 Models for games

We now link possibility correspondences with strategic games. Given a restriction $G := (G_1, \ldots, G_n)$ of the initial game H, by a **model** for G we mean a set of states Ω together with a sequence of functions $\overline{s_i} : \Omega \to G_i$, where $i \in \{1, \ldots, n\}$. We denote it by $(\Omega, \overline{s_1}, \ldots, \overline{s_n})$.

In what follows, given a function f and a subset E of its domain, we denote by $f(E)$ the range of f on E and by $f \mid E$ the restriction of f to E.

By the **standard model** $\mathcal{M}$ for G we mean the model in which

- $\Omega := G_1 \times \ldots \times G_n$ (which means that for $\omega \in \Omega$, ω_i is well-defined),
- $\overline{s_i}(\omega) := \omega_i$.

So the states of the standard model for G are exactly the joint strategies in G, and each $\overline{s_i}$ is a projection function. Since the initial game H is given, we know the payoff functions $p_1, \ldots, p_n$. So in the context of H a standard model is just an alternative way of representing a restriction of H.

Given a (not necessarily standard) model $\mathcal{M} := (\Omega, \overline{s_1}, \ldots, \overline{s_n})$ for a restriction G and a vector of events $\overline{E} = (E_1, \ldots, E_n)$ in $\mathcal{M}$ we define

$$G_{\overline{E}} := (\overline{s_1}(E_1), \ldots, \overline{s_n}(E_n))$$

and call it the **restriction of G to $\overline{E}$**. When each E_i equals E we write G_E instead of $G_{\overline{E}}$.

Finally, we extend the notion of a model for a restriction G to a **knowledge model** for G by assuming that each player i has a knowledge correspondence P_i on Ω.

3 Local and global properties

Given player i in the initial strategic game $H := (H_1, \ldots, H_n, p_1, \ldots, p_n)$ we formalize his notion of optimality using a property $\phi_i(s_i, G)$ that holds between a strategy $s_i \in G_i$ and a restriction G of H. Intuitively, $\phi_i(s_i, G)$ holds if s_i is an 'optimal' strategy for player i within the restriction G, assuming that he uses the property ϕ_i to select optimal strategies.

We distinguish between what we call 'local' and 'global' optimality. This terminology is from Apt [2007]. To assess optimality of a strategy s_i *locally* within the restriction G, it is sufficient for i to compare s_i with *only those strategies s_i' that occur in G_i*. On the other hand, to assess the optimality of s_i *globally*, player i must consider all of his strategies s_i' that occur in his strategy set H_i in the *initial game H*.

Global properties are then those in which a player's strategy is evaluated with respect to all his strategies in the initial game, whereas local properties are concerned solely with a comparison of strategies available in the restriction G. We will write ϕ^l when we refer to a local property, and ϕ^g when we refer to a global property. Here are some natural examples. We also give one example in both its local and global form in order to illustrate the distinction between them:

- $sd_i^l(s_i, G)$ that holds iff the strategy s_i of player i is not strictly dominated on G by any strategy from G_i (i.e., $\neg\exists s_i' \in G_i \; s_i' \succ_G s_i$),
- $sd_i^g(s_i, G)$ that holds iff the strategy s_i of player i is not strictly dominated on G by any strategy from H_i (i.e., $\neg\exists s_i' \in H_i \; s_i' \succ_G s_i$),
- $wd_i^l(s_i, G)$ that holds iff the strategy s_i of player i is not weakly dominated on G by any strategy from G_i (i.e., $\neg\exists s_i' \in G_i \; s_i' \succ_G^w s_i$),
- $br_i^l(s_i, G)$ that holds iff the strategy s_i of player i is a best response among G_i to some $s_{-i} \in G_{-i}$, i.e., $\exists s_{-i} \in G_{-i} \forall s_i' \in G_i p_i(s_i, s_{-i}) \geq p_i(s_i', s_{-i})$).

Each sequence of properties $\overline{\phi} := (\phi_1, \ldots, \phi_n)$ determines an operator $T_{\overline{\phi}}$ on the restrictions of H defined by

$$T_{\overline{\phi}}(G) := (G_1', \ldots, G_n'),$$

where $G := (G_1, \ldots, G_n)$ and for all $i \in \{1, \ldots, n\}$

$$G_i' := \{s_i \in G_i \mid \phi_i(s_i, G)\}.$$

Since $T_{\overline{\phi}}$ is contracting, by Note 1 it has an outcome, i.e., $T_{\overline{\phi}}^\infty$ is well-defined. $T_{\overline{\phi}}(G)$ is the result of removing from G all strategies that are not ϕ_i-optimal. So the outcome of $T_{\overline{\phi}}$ is the result of the iterated elimination of strategies that for player i are not ϕ_i-optimal, where $i \in \{1, \ldots, n\}$. In the literature, see, e.g. Lipman [1991], Chen, Long and Luo [2007] and Apt [2007], examples are given showing that for the above optimality notions in general transfinite iterations are necessary to reach the outcome.

When each property ϕ_i equals sd^l, we write T_{sd^l} instead of $T_{\overline{sd^l}}$ and similarly with other specific properties.

In the next section we assume that each player i employs some property ϕ_i to select his strategies, and we analyze the situation in which this information becomes commonly known via public announcements. To determine which strategies are then selected by the players we shall use the $T_{\overline{\phi}}$ operator.

4 Public announcements of optimality

We show that for a large class of properties ϕ_i the outcome $T_{\overline{\phi}}^{\infty}$ of the iterated elimination of strategies that for player i are not ϕ_i-optimal can be characterized by means of the concept of a *public announcement*. This approach, inspired by van Benthem [2007], applies to all global properties introduced in Section 3.

   The particular kind of "public announcement" that we will be interested in is a set of true statements, one by each player i to the effect that i will not play any strategy that is not optimal for him, according to his notion of optimality. Note that there is no strategic element to these announcements: the players simply follow a protocol from which they cannot deviate. The announcements are "public" in the sense that every other player 'hears' them as they happen.

   The iterated public announcements can be thought of as a process in which the players *learn* how the game will be played. The limit of this learning process represents the situation in which the announcements lead to no change in the model, at which point it can be said that *rationality* has been learned by all players. It is in this sense that public announcements provide alternative epistemic foundations for the outcome $T_{\overline{\phi}}^{\infty}$.

   Let us clarify first what we would like to achieve. Consider a model $\mathcal{M}$ for the initial game H. The process of iterated elimination of the strategies that are not ϕ_i-optimal, formalized by the iterated applications of the $T_{\overline{\phi}}$ operator, produces a sequence $T_{\overline{\phi}}^{\alpha}$, where α is an ordinal, of restrictions of H. We would like to mimic it on the side of the models, so that we get a corresponding sequence $\mathcal{M}^{\alpha}$ of models of these restrictions.

   To make this idea work we need to define an appropriate way of reducing models. We take care of it by letting the players repeatedly announce that they only select ϕ_i-optimal strategies. This brings us to the notions of public announcements and their effects on the models.

   Given a model $\mathcal{M} = (\Omega, \overline{s_1}, \ldots, \overline{s_n})$ for some restriction of H we define

- a ***public announcement*** by player i in a model $\mathcal{M}$ as an event E in $\mathcal{M}$,
- given a vector $\overline{E} := (E_1, \ldots, E_n)$ of public announcements by players $1, \ldots, n$ we let

$$[\overline{E}](\mathcal{M}) := (\cap_{i=1}^{n} E_i, (\overline{s_j} \,|\, \cap_{i=1}^{n} E_i)_{j \in \{1, \ldots, n\}})$$

and call it ***the effect of the public announcements of $\overline{E}$ on $\mathcal{M}$***.

Given a property ϕ_i that player i uses to select his strategies in the restriction G of H and a model $\mathcal{M} := (\Omega, \overline{s_1}, \ldots, \overline{s_n})$ for G we define $[\![\phi_i]\!]$ as the event in $\mathcal{M}$ that player i selects optimally his strategies with respect to G. Formally:

$$[\![\phi_i]\!] := \{\omega \in \Omega \mid \phi_i(\overline{s_i}(\omega), G)\}.$$

We abbreviate the vector $([\![\phi_1]\!], \ldots, [\![\phi_n]\!])$ to $[\![\overline{\phi}]\!]$.

We want now to obtain the reduction of a model $\mathcal{M}$ of G to a model $\mathcal{M}$ of $T_{\overline{\phi}}(G)$ by means of the just defined vector $[\![\overline{\phi}]\!]$ of public announcements.

The effect of the public announcements of $\overline{E}$ on a model of G should ideally be a model of the restriction $G_{\overline{E}}$. Unfortunately, this does not hold in such generality. Indeed, let the two-player game G have the strategy sets $G_1 := \{U, D\}$, $G_2 := \{L, R\}$ and consider the model $\mathcal{M}$ for G with $\Omega := \{\omega_{ul}, \omega_{dr}\}$ and the functions $\overline{s_1}$ and $\overline{s_2}$ defined by

$$\overline{s_1}(\omega_{ul}) = U, \ \overline{s_2}(\omega_{ul}) = L, \ \overline{s_1}(\omega_{dr}) = D, \ \overline{s_2}(\omega_{dr}) = R.$$

Let $\overline{E} = (\{\omega_{ul}\}, \{\omega_{dr}\})$. Then $[\overline{E}](\mathcal{M}) = \emptyset$, which is not a model of $G_{\overline{E}} = (\{U\}, \{R\})$. This simple example is depicted in Figure 1.

$$
\begin{array}{c|c|c|}
 & L & R \\
\hline
U & \omega_{ul} & \\
\hline
D & & \omega_{dr} \\
\hline
\end{array}
$$

Fig. 1. A motivating example for the use of standard models

A remedy lies in restricting one's attention to standard models. However, in order to find a faithful public announcement analogue to strategy elimination we must also narrow the concept of a public announcement as follows. A ***proper public announcement*** by player i in a standard model is a subset of $\Omega = G_1 \times \ldots \times G_n$ of the form $G_1 \times \ldots \times G_{i-1} \times G_i' \times G_{i+1} \times \ldots \times G_n$.

So a proper public announcement by a player is an event that amounts to a 'declaration' by the player that he will limit his attention to a subset of his strategies, that is, will discard the remaining strategies. So when each player makes a proper public announcement, their combined effect on the standard model is that the set of states (or equivalently, the set of joint strategies) becomes appropriately restricted. An example, which is crucial for us, of a proper public announcement in a standard model is of course $[\![\phi_i]\!]$.

The following note links in the desired way two notions we introduced. It states that the effect of the proper public announcements

of $\overline{E}$ on the standard model for G is the standard model for the restriction of $G_{\overline{E}}$.

Note 2. Let $\mathcal{M}$ be the standard model for G and $\overline{E}$ a vector of proper public announcements by players $1, \ldots, n$ in $\mathcal{M}$. Then $[\overline{E}](\mathcal{M})$ is the standard model for $G_{\overline{E}}$.

Proof. We only need to check that $\cap_{i=1}^{n} E_i$ is the set of joint strategies of the restriction $G_{\overline{E}}$. But each E_i is a proper announcement, so it is of the form $G_1 \times \ldots \times G_{i-1} \times G'_i \times G_{i+1} \times \ldots \times G_n$, where $G = (G_1, \ldots, G_n)$. So $\cap_{i=1}^{n} E_i = G'_1 \times \ldots \times G'_n$.

Moreover, each function $\overline{s_i}$ is a projection, so

$$G_{\overline{E}} = (\overline{s_1}(E_1), \ldots, \overline{s_n}(E_n)) = (G'_1, \ldots, G'_n).$$

$\square$

We also have the following observation that links the vector $[\overline{\phi}]$ of public announcements with the operator $T_{\overline{\phi}}$ of Section 3.

Note 3. Let $\mathcal{M} := (\Omega, \overline{s_1}, \ldots, \overline{s_n})$ be the standard model for G. Then

$$T_{\overline{\phi}}(G) = G_{[\overline{\phi}]}.$$

Proof. Let $G = (G_1, \ldots, G_n)$, $T_{\overline{\phi}}(G) = (G'_1, \ldots, G'_n)$ and $G_{[\overline{\phi}]} = (S''_1, \ldots, S''_n)$.

Fix $i \in \{1, \ldots, n\}$. Then we have the following string of equivalences:

$$
\begin{aligned}
& s_i \in G'_i \\
\text{iff } & s_i \in G_i \wedge \phi_i(s_i, G) \\
(\overline{s_i} \text{ is onto) iff } & s_i \in G_i \wedge \exists \omega \in \Omega \left(s_i = \overline{s_i}(\omega) \wedge \phi_i(\overline{s_i}(\omega), G) \right) \\
\text{iff } & s_i \in G_i \wedge \exists \omega \in [\phi_i] \left(s_i = \overline{s_i}(\omega) \right) \\
\text{iff } & s_i \in S''_i.
\end{aligned}
$$

$\square$

Denote now by $[\overline{\phi}]^{\infty}$ the iterated effect of the public announcements of $[\overline{\phi}]$ starting with the standard model for the initial game H. The following conclusion then relates the iterated elimination of the strategies that for player i are not ϕ_i-optimal to the iterated effects of the corresponding public announcements.

Theorem 1. $[\overline{\phi}]^{\infty}$ *is the standard model for the restriction* $T_{\overline{\phi}}^{\infty}$.

Proof. By Notes 2 and 3. $\square$

Note that in the above theorem each effect of the public announcements of $[\overline{\phi}]$ is considered on a different standard model.

5 Public announcements of rationality

The above analysis gives an account of public announcements of the *optimality* of players' strategies. We now extend this analysis to public announcements of *rationality* which takes into account players' knowledge. To this end we additionally assume for each player a knowledge correspondence $P_i : \Omega \to \mathcal{P}(\Omega)$.

We define then the event of player i being ϕ_i-**rational** in the restriction G as

$$\langle \phi_i \rangle := \{\omega \in \Omega \mid \phi_i(\overline{s_i}(\omega), G_{P_i(\omega)})\}.$$

Note that when player i knows that the state is in $P_i(\omega)$, the restriction $G_{P_i(\omega)}$ represents his knowledge about the players' strategies. That is, $G_{P_i(\omega)}$ is the game he knows to be relevant to his choice. Hence $\phi_i(\overline{s_i}(\omega), G_{P_i(\omega)})$ captures the idea that if player i uses ϕ_i to select his optimal strategy in the game he considers relevant, then in the state ω he indeed acts 'rationally'.

Again we abbreviate $(\langle \phi_1 \rangle, \ldots, \langle \phi_n \rangle)$ to $\langle \overline{\phi} \rangle$. Note that $\langle \overline{\phi} \rangle$ depends on the underlying knowledge model $(\Omega, \overline{s_1}, \ldots, \overline{s_n}, P_1, \ldots, P_n)$ and on G.

We extend the definition of the effect of the public announcements $\overline{E} := (E_1, \ldots, E_n)$ to knowledge models in the natural way, by restricting each possibility correspondence to the intersection of the events in $\overline{E}$:

$$[\overline{E}](\mathcal{M}, P_1, \ldots, P_n) = ([\overline{E}]\mathcal{M}, P_1 \mid \cap_{i=1}^{n} E_i, \ldots, P_n \mid \cap_{i=1}^{n} E_i).$$

This definition is in the same spirit as in Plaza [1989] and in Osborne and Rubinstein [1994, page 72], where it is used in the analysis of the puzzle of the hats.

We aim to find a class of knowledge models for which, under a mild restriction on the properties ϕ_i, $\langle \overline{\phi} \rangle^\infty$, the iterated effect of the public announcements of $\langle \overline{\phi} \rangle$ starting with the standard knowledge model for the initial game H, will be the standard knowledge model for $T_{\overline{\phi}}^\infty$. We will therefore use a natural choice of possibility correspondences, which we call the **standard possibility correspondences**:

$$P_i(\omega) = \{\omega' \in \Omega \mid \overline{s_i}(\omega) = \overline{s_i}(\omega')\}.$$

In particular, for the standard model

$$P_i(\omega) := \{\omega' \mid \omega_i = \omega_i'\}$$

By the **standard knowledge model** for a restriction G we now mean the standard model for G endowed with the standard possibility correspondences.

The following observation holds.

Note 4. Consider the standard knowledge model $(\Omega, \overline{s_1}, \ldots, \overline{s_n}, P_1, \ldots, P_n)$ for a restriction $G := (G_1, \ldots, G_n)$ of H and a state $\omega \in \Omega$. Then for all $\omega \in \Omega$

$$G_{P_i(\omega)} = (G_1, \ldots, G_{i-1}, \{\omega_i\}, G_{i+1}, \ldots, G_n).$$

Proof. Immediate by the fact that in the standard knowledge model for each possibility correspondence we have $P_i(\omega) = \{\omega' \in \Omega \mid \omega'_i = \omega_i\}$. $\qquad\square$

Intuitively, this observation states that in each state of a standard knowledge model each player knows his own choice of strategy but knows nothing about the strategies of the other players. So standard possibility correspondences represent the knowledge of each player after he has privately selected his strategy but no information between the players has been exchanged. It is in that sense that the standard knowledge models are natural. In van Benthem [2007] in effect only such models are considered.

A large class of properties ϕ_i satisfy the following restriction:

A For all $(G_1, \ldots, G_n)$ and G'_i
$\phi_i(s_i, (G_i, G_{-i})) \leftrightarrow \phi_i(s_i, (G'_i, G_{-i}))$.

That restriction on the properties ϕ_i is sufficient to obtain the following analogue of Theorem 1 for the case of public announcements of *rationality*.

Theorem 2. *Suppose that each property $\phi_1, \ldots, \phi_n$ satisfies **A**. Then $\langle \overline{\phi} \rangle^\infty$ is the standard knowledge model for the restriction $T^\infty_{\overline{\phi}}$.*

Proof. Notice that it suffices to prove for each restriction G the following statement for each i:

$$\forall \omega \in \Omega \; (\phi_i(\omega_i, G) \leftrightarrow \phi_i(\omega_i, G_{P_i(\omega)})). \tag{1}$$

Indeed, (1) entails that $\langle \phi_i \rangle = [\![\phi_i]\!]$, in which case the result follows from Theorem 1 and the observation that the possibility correspondences are restricted in the appropriate way.

But (1) is a direct consequence of the assumption of **A** and of Note 4. $\qquad\square$

To see the consequences of the above result note that **A** holds for each global property sd^g_i, wd^g_i, and br^g_i introduced in Section 3.

For each ϕ_i equal sd^g_i and finite games Theorem 2 boils down to Theorem 7 in van Benthem [2007]. The corresponding result for each ϕ_i equal to br^g_i and finite games is mentioned at the end of Section 5.4 of that paper.

It is important to note that the above Theorem does not hold for the corresponding local properties sd_i^l, wd_i^l, or br_i^l introduced in Section 3. Indeed, for each such property ϕ_i we have $\phi_i(\omega_i, G)$ for each state ω_i and restriction G in which player i has just one strategy. So by Note 4 $\phi_i(\omega_i, G_{P_i(\omega)})$ holds for each state ω and restriction G. Consequently $\langle \overline{\phi} \rangle = \Omega$. So when each ϕ_i is a local property listed above, $\langle \overline{\phi} \rangle$ is an identity operator on the standard knowledge models, that is $\langle \overline{\phi} \rangle^\infty$ is the standard knowledge model for the initial game H and not $T_{\overline{\phi}}^\infty$.

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